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Various Shape Functions of Magnetic Curves

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ABSTRACT

Magnetic curves are particle trajectories traced under the influence of magnetic field. It encompasses a huge family of monotonic varying curvature curves. Recently, magnetic curves with linear logarithmic curvature graph (LCG) has been derived and denoted as log-aesthetic magnetic curve (LMC), whose characteristics are similar to that of log-aesthetic curves (LAC). This paper elaborates aesthetic shapes through analyzing the particle charge function $q(t)$ of general representation of magnetic curves which dictates the monotonicity of curvature profile. This paper also proves that LMC or the generalized version of LMC is not the only representation of the family of magnetic curve with monotonic curvature profile. The investigation of the fairness of planar magnetic curves uses conventional shape interrogation methods and LCG. The result provides an insight to control the shape function $q(t)$ for the generation of fair magnetic curves and elucidate their properties for practical design purposes.

Keywords: Aesthetic curves, spirals.

1. Introduction

Farin et. al., 2002, defined Computer Aided Graphic Design (CAGD) as the construction and representation of free-form curves, surfaces or volumes. Although there are various types of developed curves in CAGD, few meets the high aesthetic need in automobile and aircraft designs due to lack of smoothness and controllability. Conventional curves used in such industries are Bézier and Non-Uniform Rational B-Spline (NURBS) curves. These curves have drawbacks in their curvature formulas because natural spirals have much simpler formulas than those of Bézier and NURBS (Gobithaasan, 2013). Furthermore, the obvious oscillation in their curvature plot makes it less visually pleasing compared to aesthetic curves and more effort is needed to reduce the oscillation of their curvatures.

Miura, 2006, introduced a general formula of aesthetic curves which is later renamed as log-aesthetic curve (LAC). LAC is described by Yoshida and Saito, 2006, as a curve with linear Logarithmic Curvature Graph (LCG) and is deemed as high quality curves. They also classified and listed members of the aesthetic curves in standard form which includes clothoid, logarithmic spiral and circle involute and studied the relationship between the slopes of straight line of their respective LCG and their shapes. They showed that the shape of the curve changes as the value of LCG slope varies. Later, Gobithaasan and Miura proposed the general form of the LAC's formula known as the Generalized LAC (GLAC). It has an extra degree of freedom compared to LAC which results in increased flexibility. Recently, a comprehensive study of aesthetic curves for CAGD is elaborated in Miura and Gobithaasan, 2014.

Xu and Mould, 2009, proposed a magnetic curve for artistic designs. It is a particle tracing method that produces curves of constantly varying curvature. These curves are usually spirals or helices. They further demonstrated its applications in computer graphics via rendering graphics such as stylized trees, hairs, water and fire. A review of the formulation of magnetic curve and deriving magnetic curves with constant LCG, denoted as Log-aesthetic Magnetic Curves (LMC) has been done by Wo *et. al.*, 2014, recently. Since the particle charge of magnetic curve is an arbitrary real function, various fair curves with monotone curvature can be created for design purposes. The main purpose of this paper is to determine the particle charge functions $q(t)$ and their practicality for CAD systems such that a wider range of monotone curvature curves can be used for design purposes.

2. Fundamentals of Magnetic Curves

A magnetic curve is the resulting trajectory of the particle under the influence of Lorentz Force (Xu and Mould, 2009):

$$\vec{F} = m\frac{d\vec{v}}{dt} = q(\vec{v} \times \vec{B}), \tag{1}$$

where m, q, $\vec{v}$ and $\vec{B}$ are the particle mass, charge and velocity vector, and magnetic induction vector respectively. A parametric magnetic curve in $\mathbf{R}^3$ space is derived as

$$C(t) = \int_{t_0}^{t} (\, v_\perp \cos\theta(t)\,, v_\perp \ \sin\theta(t)\,, v_\parallel)\; dt. \tag{2}$$

The parameter t is a time parameter while $v_\perp$ and $v_\parallel$ are magnitudes of the components of $\vec{v}$, which are perpendicular and parallel to $\vec{B}$ respectively (see Figure 1). The function $\theta(t)$ is the tangential angle of (2), which can be written as $\theta(t) = \int_{t_0}^{t} \omega(t)\; dt$, where

$$\omega(t) = \frac{B}{m} q(t). \tag{3}$$

The term t_{org} denotes the time when the position of the particle is at the origin (0,0) with tangent vector $\langle v_\perp, 0\rangle$. Equation (3) is the gyro-frequency of the trajectory whereas $q(t)$ is the particle charge function and can be any arbitrary real function. B is the magnitude of magnetic induction vector $\vec{B}$. We set $v_\parallel = 0$ and fix $\vec{B}$ such that $\vec{B}$ is parallel to the z-axis. This is done in effort to limit the curve $C(t)$ to the x-y plane. Thus $C'(t) = \overrightarrow{v_\perp}$. Figure 2 depicts a magnetic curve along with its velocity vector and tangent angle.

The radius of gyration, or also known as radius of curvature of a planar magnetic curve is given as:

$$\rho(t) = \frac{s'(t)}{\theta'(t)} = \frac{v_\perp}{B|q(t)|}. \tag{4}$$

Generally, magnetic curve have the arc length:

$$s(t) = v_\perp (t - t_0) \tag{5}$$

Thus, it is arc length-parameterized when $v_\perp = 1$. The sign of the function $q(t)$ governs the direction of the particle acceleration. For example, a positively signed $q(t)$ produces a magnetic curve which turns towards the left. If the sign of $q(t)$ changes to negative, the curve turns toward the right. When $q(t)$ is a constant, the curve is a circle in $\mathbf{R}^2$ or a helix in $\mathbf{R}^3$ space.

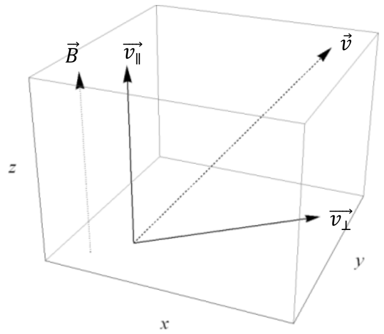


Figure 1: The direction of $\vec{B}$, $\vec{v}$, and two components of $\vec{v}$; $\overrightarrow{v_\perp}$ and $\overrightarrow{v_\parallel}$.

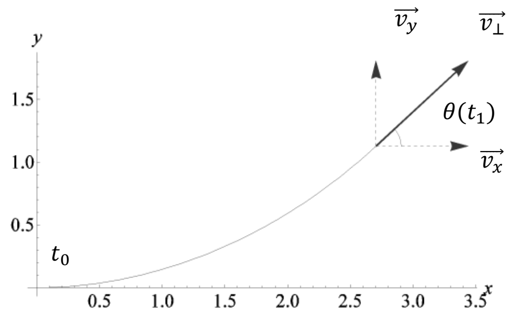


Figure 2: Magnetic curve, its velocity vector $\overrightarrow{v_\perp} = \langle v_x, v_y \rangle$ and tangential angle $\theta(t_1)$.

3. LCG Gradient of Magnetic Curve

LCG (Gobithaasan and Miura, 2014; Yoshida *et. al.*, 2010) is the analytical representation of Logarithmic Distribution Diagram of Curvature (LDDC) which represents the relationship between interval of radius of curvature and arc length frequency. The equation of LCG is given below:

$$LCG(t) = \left\{ \log \rho(t), \log \frac{\rho(t)s'(t)}{\rho'(t)} \right\}. \tag{6}$$

The gradient of LCG is

$$\lambda(t) = 1 + \frac{\rho(t)}{\rho'(t)^2}\left(\frac{\rho'(t)s''(t)}{s'(t)} - \rho''(t) \right). \tag{7}$$

A curve is classified as LAC if its LCG gradient is a constant, denoted by α . In general, magnetic curve has the following LCG gradient:

$$\lambda(t)=\frac{q(t)q''(t)}{q'(t)^2}-1, \tag{8}$$

after substituting (4) and (5) into (7). Hence the particle charge function $q(t)$ affects directly both radius of curvature and LCG gradient.

4. Magnetic Curve with Monotone Curvature

It is discussed in the previous section that $q(t)$ has direct control over a magnetic curve's curvature profile. Thus, monotonicity of the function $q(t)$ guarantees the monotonicity of the curvature profile, $(t)=\frac{1}{q(t)}$. For generality, $q(t)$ is set to be $g(t)+l\left(\frac{v_\perp}{B}\right)$ where $l\in\mathbf{R}$ is the y- interception of the curvature profile and $g(t)$ as an arbitrary monotone function:

$$\kappa(t)=\frac{B}{v_\perp}q(t)=\frac{B}{v_\perp}g(t)+l. \tag{9}$$

Therefore, the slope of the curvature can be manipulated via parameters B and $v_\perp$, while $\kappa(t)$ can be shifted by manipulating l. An inflection point $(\kappa(t)=0)$ may or may not occur depending on the value of l.

Three categories of functions for $q(t)$ are studied in this chapter. These are the elementary algebraic (polynomials), transcendental (exponential, hyperbolic, logarithms, trigonometric and power functions) and two types of conic functions (ellipse and parabola).

4.1 Polynomials

Magnetic curve with polynomial curvature functions can be referred as polynomial spirals as in Delingette *et al.*, 1991, since they are of the same form. These spirals are used to generate trajectories of G^2-continuity while minimizing the path energy. These curves have a high degree of freedom but the monotonicity of the curvature is not guaranteed for polynomials of degree three and above. Hence, calling these curves as polynomial spiral is rather misleading. In order to guarantee curvature monotonicity, the derivative of the polynomial curvature must be inspected and manipulated such that $\kappa'(t)\neq 0$ for all $t\in\mathbf{R}$. Figure 3 shows an example of magnetic spiral of monotone polynomial curvature, $q(t) = 0.375\, t^2 + \frac{2}{3}\, t^3 + 0.125\, t^4$.

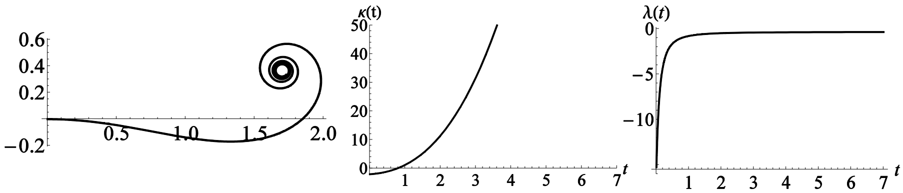


Figure 3: A magnetic curve (left) with parabolic curvature function (middle), and LCG gradient function (right). The inputs are $l = -2; B = 0.2;\ v_{\perp} = 1; t \in (0,7]$.

The magnetic curve in
Figure 3 has an inflection point near $t = 1$ which is also the real root of the polynomial equation $\kappa(t) = 0.2\left(0.375\, t^2 + \frac{2}{3}\, t^3 + 0.125\, t^4\right) - 2$ for $t \geq 0$. The inflection point occurs at $t = 0$ when $l = 0$. If $l > 0$, then the curve will be a C-shaped curve.

4.2 Elementary Transcendental Function

When $l = 0$, the curves of exponential and power curvature functions forms the family of LMC with constant LCG gradient, where

$$q_{GLMC}(t) = \begin{cases} t^{-\beta}, & \beta \in \mathbf{R}, \\ e^t, & otherwise. \end{cases} \tag{10}$$

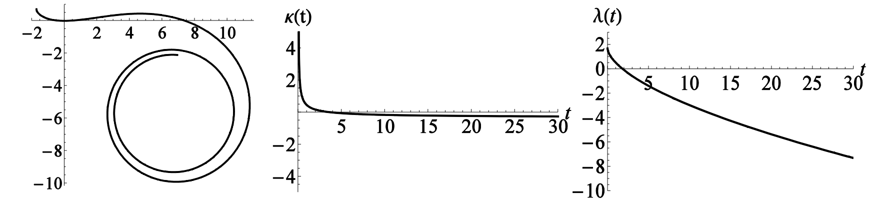
However, $q_{GLMC}(t)$ stated below:

$$q_{GLMC}(t) = \begin{cases} t^{-\beta} + \dfrac{v_{\perp}}{B} l, & \beta \in \mathbf{R}, \\ e^t + \dfrac{v_{\perp}}{B} l, & otherwise. \end{cases} \tag{11}$$

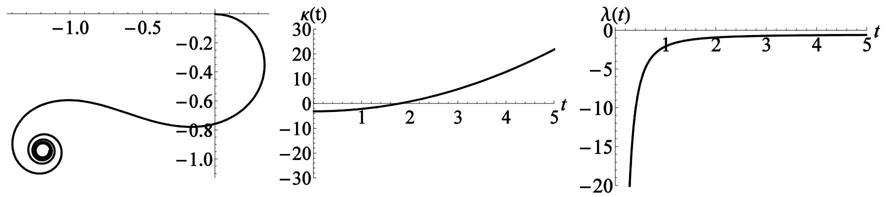
is a general case of (10) where the curve traced with (11) is denoted as generalized LMC (GLMC). The bound for the parameter $t \in \mathbf{R}^3$ for the curve of (11) is $t \in (0, \infty)$ for $\beta > 0$, $t \in [0, \infty)$ for $B < 0$ and $t \in (-\infty, \infty)$ otherwise. The LCG of (11) is given below:

$$\lambda_{GLMC}(t) = \begin{cases} \frac{B + t^{\beta} v_{\perp} l (1+\beta)}{B\,\beta}, & \beta \neq 0, \\ e^t \frac{v_{\perp} l}{B}, & otherwise. \end{cases} \tag{12}$$

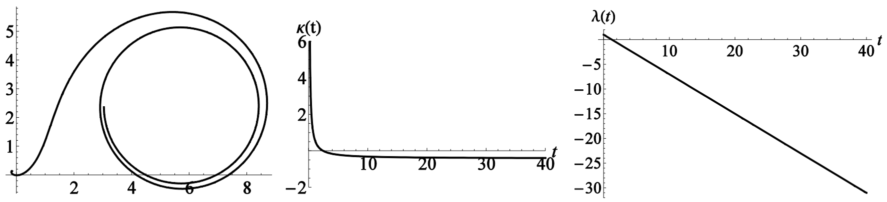
Examples of these curves are shown in Figure 4.



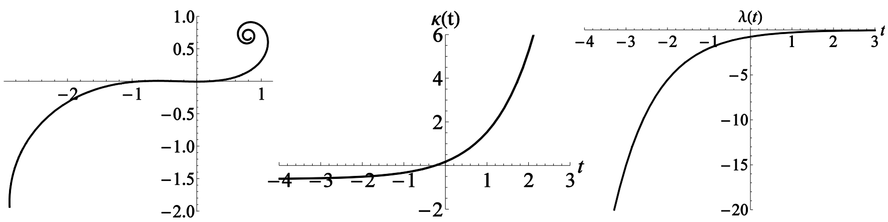
(a)



(b)



(c)



(d)

Figure 4: GLMC (left) with curvature function (middle) and LCG gradient function (right). The inputs are shown in TABLE 1.

TABLE 1: Inputs for Figure 4

Figure	$q_{GLMC}(t)$	β	B	$v_{\perp}$	l	t_0
(a)	$\beta \in \mathbf{R}$	0.6	0.8	1.0	-0.35	2.0
(b)	$\beta \in \mathbf{R}$	-2.0	1.0	1.0	-3.00	0.0
(c)	$\beta \in \mathbf{R}$	1.0	1.0	1.0	-0.40	0.3
(d)	$otherwise$	-	0.8	1.0	-0.60	0.0

Setting $l < 0$ allows GLMC to have inflection points when their LMC counterparts could not. Examples are given in Figure 4 (a), (c) and (d). Inflection point of a GLMC with $\beta > 0$ approaches $t = 0$ as l decreases. It moves further away from the origin as l decreases for $\beta < 0$ and $1/\beta = 0$. Since $\kappa(t)$ is monotonic, there can be only one inflection point on a GLMC.

Other monotone elementary transcendental functions that can be substituted as particle charge function $g(t)$ are logarithms, hyperbolic and inverse tangent functions. Trigonometric functions are not studied in this paper due to its periodic nature, albeit the bounds of t can be set such that the functions are strictly monotone in their respective bounds. The boundary for t is $(0, \infty)$ for $(t) = \ln t$, the curve of logarithmic curvature profile. An example is given in Figure 5 5.

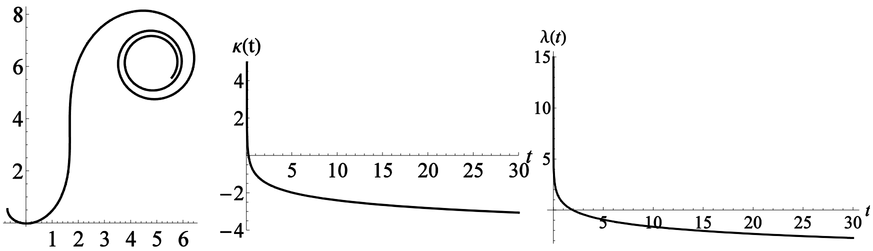


Figure 5: A magnetic curve (left) with curvature function (middle) and LCG gradient function (right). The inputs are $l = -1; B = 1.2;\ v_{\perp} = 2; t \in (0,10]$, with $t_0 = 1$.

Hyperbolic functions are symmetric, thus they are monotonic for $t > 0$ or $t < 0$. The bounds for t are $(-\infty, 0) \cup (0, \infty)$ for hyperbolic cosecant and cotangent and $(-\infty, \infty)$ for hyperbolic sine. The example shown in Figure 6 has the function $g(t) = \sinh t$.

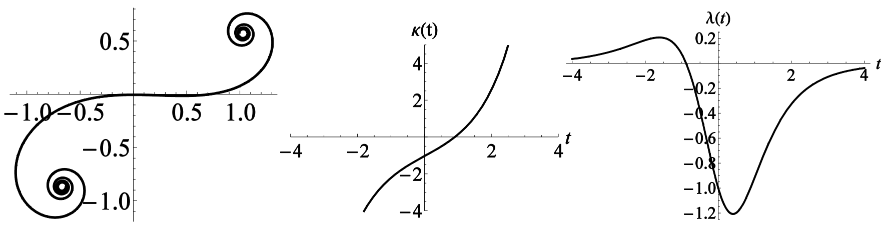


Figure 6: A magnetic curve (left) with hyperbolic sine curvature function (middle) and LCG gradient function (right). The inputs are are $l = -0.8; B = 1.2;\ v_{\perp} = 1; t \in [-4,4]$, with $t_0 = 0.5$.

The inflection point of the magnetic curve in Figure 6 occurs near $t = 1$. There are three pieces of LCG for the curve, the first one increasing, reaches a maximum near $t = -2.5$, then decrease to a minimum near $t = 0.5$ before increasing again. This indicates that the curve is initially divergent at $t = -4$, gradually converging afterwards and becomes divergent again.

4.3 Ellipse and Parabola

Half ellipse and parabola are monotone functions for either $t \geq 0$ or $t \leq 0$. These functions are symmetrical, thus either the upper half or the lower half of the ellipse or parabola can be assigned as the function $g(t)$ to produce monotone curvature. The resulting elliptic and parabolic curvature functions are given below

$$\kappa_{ellp}(t) = \frac{B}{v_{\perp}} \sqrt{\frac{b^2(1 - t^2)}{a^2}} + l \tag{13}$$

$$\kappa_{para}(t) = \frac{B}{v_{\perp}} \sqrt{\frac{b^2(1 + t^2)}{a^2}} + l \tag{14}$$

The parameter t for parabola is not bounded, however it is bounded by $[-a, a]$. The curves of (13) and (14) are presented in Figure 7 and Figure 8 along with their respective curvature and LCG gradient plots.

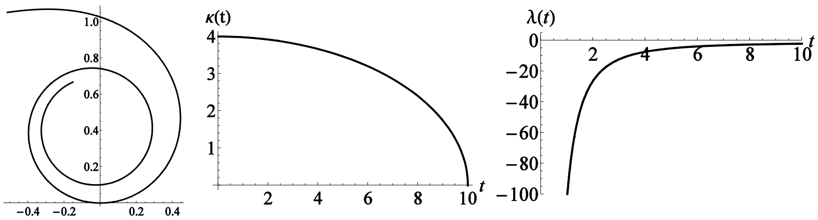


Figure 7: A magnetic curve (left) with elliptic curvature function (middle), and LCG gradient function (right). The inputs are $a = 10; b = 4;\ l = 0; B = 1;\ v_{\perp} = 1; t \in [0,10]$, with $t_0 = 8$.

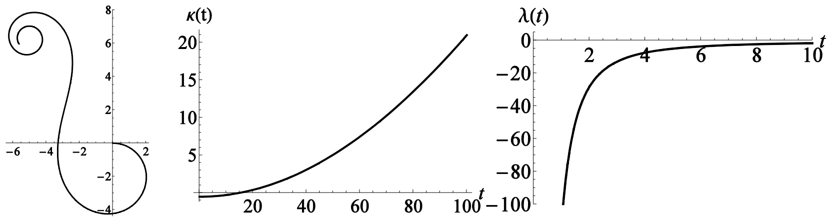


Figure 8: A magnetic curve (left) with parabolic curvature function (middle), and LCG gradient function (right). The inputs are $a = 214; b = 207;\ l = -0.5; B = 1;\ v_{\perp} = 1; t \in [0,30]$.

5. Conclusion

It can be concluded that the LCG gradient for magnetic curves with monotone curvatures do not consistently follow a pattern by inspecting the $q(t)$ functions in the previous section. Some of the particle charge function generates an almost linear LCG while some shows some obvious oscillation in the LCG gradient function. Beside generalized LMC, magnetic curve with polynomial and elliptic curvature are good candidates for designing fair shapes as they have more degree of freedom compared to the other functions discussed in this section. It is shown that there are various spirals that can be generated with elementary monotonic functions and proven that LMC is a part of a bigger family of magnetic curves with monotone curvature. In general, the monotonicity of $q(t)$ generates various magnetic spirals that can be classified as quasi-aesthetic curves (Yoshida and Saito, 2007).

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